The Horn Tarski problem

Egbert Thümmel

May 11, 2012

Abstract

In 1948 Horn and Tarski asked whether the notions of a σ-finite cc and a σ-bounded cc ordering are equivalent. We give a negative answer to this question.

When analyzing Boolean algebras carrying a measure, Horn and Tarski [HT48] defined the following two notions:

Definition 1. An ordering $\mathbb{P}$ is called

(i) *σ-bounded cc* if $\mathbb{P} = \bigcup_{n\in\omega} P_n$, where each P_n has the $n+2$-cc.

(ii) *σ-finite cc* if $\mathbb{P} = \bigcup_{n\in\omega} P_n$, where each P_n has the ω-cc.

Here an ordering or its subset has the κ-cc (κ-chain condition) for a cardinal κ if it contains no antichain (set of pairwise orthogonal elements) of size κ.

Clearly, any σ-bounded cc ordering is σ-finite cc (and both are ω_1-cc - also called ccc). Horn and Tarski asked whether these two classes coincide:

Problem Horn Tarski 1948 [HT48] *Is every σ-finite cc ordering also σ-bounded cc?*

There is a standard way how to map an ordering densely into a complete Boolean algebra. This mapping preserves our two properties. The Horn Tarski problem can therefore be formulated in terms of Boolean algebras as well. It is easy to see that a Boolean algebra carrying a strictly positive measure is σ-bounded cc (take as P_n the set of elements of measure at least $1/n$). If the Boolean algebra carries only a strictly positive exhaustive submeasure this property could get lost, but still the Boolean algebra will be σ-finite cc (take the same P_n). The question, whether any Boolean algebra carrying a strictly positive exhaustive submeasure carries also a strictly positive measure, is one formulation of the famous Control measure problem. It was therefore expected that an anticipated negative solution of this problem will give also a counterexample to the Horn Tarski problem. But when such an example solving the Control measure problem was constructed by M. Talagrand ([Tal08]) it turned out that it is even σ-bounded cc, so the Horn Tarski problem remained open. We will construct here a counterexample to the Horn Tarski problem.

Theorem 1. *There exists an ordering which is σ-finite cc but not σ-bounded cc.*

The technique used in the construction appeared first in [Tod91] and is further developed in [BPT] :

For a subset F of a topological space X let F^d denote the set of all accumulation points.

Definition 2. For a topological space X the Todorcevic ordering $\mathbb{T}(X)$ is the set of all subsets F of the space which are a finite union of converging sequences including their limit points. The order relation is defined by such extensions which preserve isolated points, i.e. $F_1 \leq F_2$ if $F_1 \supseteq F_2$ and $F_1^d \cap F_2 = F_2^d$.

We start with the set $T = \bigcup_{\alpha<\omega_1} {}^{\alpha+1}\omega$. This set is made into a tree by the order of inclusion. We will extend the order of the tree into a linear one. Define an equivalence relation for the elements of the tree such that two elements are equivalent if they have the same predecessors, i.e. $s \sim t$ if $\forall r \in T(r \subset s \leftrightarrow r \subset t)$. Equip every equivalence class under $\sim$ with an order $\preceq$ of type ω^*, i.e. ω ordered in the reverse. Then we can define the lexicographical order $\leq$ on T by $s < t$ if either $s \subset t$ or there are $s' \subseteq s$ and $t' \subseteq t$ such that $s' \sim t'$ and $s' \prec t'$. Take the interval topology $\tau_{\leq}$ on T. We apply the operator $\mathbb{T}$ on this linearly ordered topological space $(T, \tau_{\leq})$ to obtain the Todorcevic ordering $\mathbb{P} = \mathbb{T}(T)$. This will be the example which proves the theorem:

Claim 1. *$\mathbb{P}$ is not σ-bounded cc.*

Proof. Assume by contradiction that $\mathbb{P} = \bigcup_{n\in\omega} P_n$, with each P_n being $n+2$-cc, witnesses that $\mathbb{P}$ is σ-bounded cc. For $n < \omega$ define functions $f_n : T \longrightarrow n+2$ such that $f_n(s)$ is the maximal length of an antichain which is a subset of the set $P_n(s) = \{F \in P_n : \exists t \in F^d(t \supseteq s)\}$. The function f_n is decreasing with respect to $\subseteq$. It follows that for any $s \in T$ there is an $s' \supseteq s$ such that $f_n(s') = f_n(t)$ for all $t \supseteq s'$. We find a increasing (with respect to $\subseteq$) sequence $\{s_n\}$ such that $f_n(s_n) = f_n(t)$ for all $t \supseteq s_n$. For an arbitrary $s \in T$ with $s \supset \bigcup_{n<\omega} s_n$ we have therefore $f_n(s) = f_n(t)$ for all $t \supseteq s$ and $n < \omega$. Fix such an s and let $f(n) = f_n(s)$. For $n < \omega$ choose in $P_n(s^\frown n)$ an antichain $\{F_{n,i}\}_{i<f(n)}$ and $t_{n,i} \supseteq s^\frown n$ such that $t_{n,i} \in (F_{n,i})^d$ for $i < f(n)$. Then $\{t_{n,i}\}_{n<\omega, i<f(n)}$ converges to s (if not finite) and so does $\{s^\frown n\}_{n<\omega}$, i.e. $F = \{t_{n,i}\}_{n<\omega,i<f(n)} \cup \{s^\frown n\}_{n<\omega} \cup \{s\} \in \mathbb{P}$. Notice that F is orthogonal to all $F_{n,i}$ for $n < \omega$ and $i < f(n)$ since $t_{n,i}$ is isolated in F and an accumulation point in $F_{n,i}$. But F has to be contained in some P_n, hence $\{F_{n,i}\}_{i<f(n)} \cup \{F\}$ is an antichain in $P_n(s)$ and therefore $f_n(s) \geq f(n) + 1$, a contradiction. □

Claim 2. *$\mathbb{P}$ is σ-finite cc.*

Proof. We argue in the order $\leq$. The set $\{s^\frown k\}_{k<\omega}$ is a decreasing sequence with infimum s. We can therefore for any $F \in \mathbb{P}$ fix a $k(F) < \omega$ such that, for $s \in F^d$, the open intervals $(s, s^\frown k(F))$ are disjoint from F^d. No increasing sequence of $(T, \leq)$ has a supremum. This means that any sequence which converges to s is above s with the possible exception of finitely many elements. Therefore $R(F) = F \backslash \big(\bigcup_{s\in F^d}(s, s^\frown k(F)) \cup F^d\big)$ is finite. Let

$$P_{k,n,m} = \{F \in \mathbb{P} \quad : \quad k(F) = k \ \& \ |F^d| = n \ \& \ |R(F)| = m\}.$$

Surely $\mathbb{P} = \bigcup_{k,n,m<\omega} P_{k,n,m}$. We have to show that all $P_{k,n,m}$'s are finite cc. Assume by contradiction that $\{F_i\}_{i<\omega} \subset P_{\bar{k},\bar{n},\bar{m}}$ is an infinite antichain for some fixed $\bar{k}, \bar{n}, \bar{m}$. Let $(F_i)^d = \{s_i^n\}_{n<\bar{n}}$ and $R(F_i) = \{r_i^m\}_{m<\bar{m}}$ be enumerated and put $F_i^n = F_i \cap (s_i^n, s_i^{n\frown}\bar{k})$. Then F_i^n is a sequence with limit s_i^n and $F_i \backslash (F_i)^d = \bigcup_{n<\bar{n}} F_i^n \cup \{r_i^m\}_{m<\bar{m}}$ is the set of isolated points of F_i. We say that $\{i, j\} \in [\omega]^2$, $i < j$, has color

$$\begin{array}{ll} (1, n, n') & \text{if } s_i^n \in F_j^{n'} \\ (2, n, m) & \text{if } s_i^n = r_j^m \\ (3, n, n') & \text{if } s_j^n \in F_i^{n'} \\ (4, n, m) & \text{if } s_j^n = r_i^m \end{array}$$

for $n, n' < \bar{n}$ and $m < \bar{m}$. Since $\{F_i\}_{i<\omega}$ was assumed to be an antichain, there must be for any $\{i, j\} \in [\omega]^2$ a point which is isolated in F_i and not isolated in F_j or vice versa, i.e. any pair $\{i, j\}$ obtains at least one color. Ramsey's theorem asserts that there must be an infinite subset of ω homogeneous in one color. For notational convenience, we assume that ω itself is this homogeneous set. We are going to derive a contradiction for each of the colors.

1. *ω is homogeneous in color $(1, n, n')$.*

Note that $s \in (t, t^\frown\bar{k})$ implies $s \supset t$ and $(s, s^\frown\bar{k}) \subset (t, t^\frown\bar{k})$.

Homogeneity in color $(1, n, n')$ implies $s_i^n \in F_j^{n'} \subseteq (s_j^{n'}, s_j^{n'\frown}\bar{k})$, i.e. $s_i^n \supset s_j^{n'}$ for all $i < j$. We have $s_{i-1}^n \supset s_i^{n'}, s_{i+1}^{n'}$, hence $s_i^{n'} \subseteq s_{i+1}^{n'}$ or $s_i^{n'} \supset s_{i+1}^{n'}$. Consider the first case. The order $\leq$ is stronger than $\subseteq$, therefore $s_i^{n'} \leq s_{i+1}^{n'} < s_{i-1}^n \in F_i^{n'} \subseteq (s_i^{n'}, s_i^{n'\frown}\bar{k})$. The latter is an interval, hence $s_{i+1}^{n'} = s_i^{n'}$ or $s_{i+1}^{n'} \in (s_i^{n'}, s_i^{n'\frown}\bar{k})$, therefore $(s_{i+1}^{n'}, s_{i+1}^{n'\frown}\bar{k}) \subseteq (s_i^{n'}, s_i^{n'\frown}\bar{k})$. But $s_i^n \notin (s_i^{n'}, s_i^{n'\frown}\bar{k})$. This follows from the definition of $\bar{k} = k(F_i)$ at the beginning of the proof. On the other hand, $s_i^n \in F_{i+1}^{n'} \subseteq (s_{i+1}^{n'}, s_{i+1}^{n'\frown}\bar{k})$ by homogeneity - a contradiction. So the second case $s_i^{n'} \supset s_{i+1}^{n'}$ must hold for all $i < \omega$, i.e. the $s_i^{n'}$'s are a strictly decreasing sequence in the tree T, again a contradiction.

2. *ω is homogeneous in color $(2, n, m)$.*

From $s_1^n = r_2^m$ and $s_0^n = r_2^m$ and $s_0^n = r_1^m$ (homogeneity in color $(2, n, m)$) we conclude $s_1^n = r_1^m$ - a contradiction since s_1^n is an accumulation point in F_1 and r_1^m is isolated in F_1.

3. *ω is homogeneous in color $(3, n, n')$.*

Assume that there are $i < j$ such that $s_i^n = s_j^n$. Then $s_i^n = s_j^n \in F_i^{n'}$, but s_i^n is an accumulation point of F_i whereas $F_i^{n'}$ contains only isolated points of F_i - a contradiction. So the s_j^n's are pairwise different for $j < \omega$. Homogeneity in color $(3, n, n')$ implies that all s_j^n, $j > 0$, are in $F_0^{n'}$, the set $\{s_j^n\}_{j=1}^{\omega}$ therefore converges to $s_0^{n'}$. By the same argument, we obtain that $\{s_j^n\}_{j=2}^{\omega}$ converges to $s_1^{n'}$, hence $s_0^{n'} = s_1^{n'}$. Again by homogeneity $s_1^n \in F_0^{n'} \subseteq (s_0^{n'}, s_0^{n'\frown}\bar{k}) = (s_1^{n'}, s_1^{n'\frown}\bar{k})$ - a contradiction since $s_1^n \notin (s_1^{n'}, s_1^{n'\frown}\bar{k})$ by definition of $\bar{k} = k(F_1)$.

4. *ω is homogeneous in color $(4, n, m)$.*

The same as color $(2, n, m)$.

For all the colors we obtained a contradiction, so an infinite antichain cannot exist.

□

References

[BPT] B. Balcar, T. Pazák, and E. Thümmel. On Todorcevic orderings. *In preparation.*

[HT48] A. Horn and A. Tarski. Measures in Boolean algebras. *Trans. Amer. Math. Soc.*, 64:467–497, 1948.

[Tal08] M. Talagrand. Maharam's problem. *Ann. of Math.*, 168(3):981–1009, 2008.

[Tod91] S. Todorčević. Two examples of Borel partially ordered sets with the countable chain condition. *Proc. Amer. Math. Soc.*, 112(4):1125–1128, 1991.